

# Contents

---

<b>1</b>	<b>Linear Algebra</b>	<b>1</b>
1.1	Vectors and Linear Spaces . . . . .	1
1.1.1	Subspaces . . . . .	2
1.1.2	Length, Distance, and Alignment . . . . .	2
1.1.3	Linear Independence and Bases . . . . .	5
1.2	Linear Operators and Matrices . . . . .	6
1.2.1	Addition and Multiplication of Matrices . . . . .	7
1.2.2	Transpose and Adjoint . . . . .	10
1.2.3	Einstein Summation Convention . . . . .	11
1.2.4	Gram-Schmidt Orthogonalization and the QR De- composition . . . . .	12
1.2.5	The Outer Product, Dyads, and Projection Operators	13
1.2.6	Partitioned Matrices and Matrix Operations . . . . .	14
1.3	Systems of Linear Algebraic Equations . . . . .	16
1.3.1	Introduction to Existence and Uniqueness . . . . .	16
1.3.2	Solving $Ax = b$ : LU Decomposition . . . . .	18
1.3.3	The Determinant . . . . .	20
1.3.4	Rank of a Matrix . . . . .	20
1.3.5	Range Space and Null Space of a Matrix . . . . .	22
1.3.6	Existence and Uniqueness in Terms of Rank and Null Space . . . . .	23
1.3.7	Least-Squares Solution . . . . .	24
1.3.8	Minimum Norm Solution . . . . .	29
1.3.9	Rank, Nullity, and the Buckingham Pi Theorem . . . . .	30
1.3.10	Nonlinear Algebraic Equations: the Newton-Raphson Method . . . . .	32
1.3.11	Linear Coordinate Transformations . . . . .	34
1.4	The Algebraic Eigenvalue Problem . . . . .	35
1.4.1	Introduction . . . . .	35
1.4.2	Self-Adjoint Matrices . . . . .	37
1.4.3	General (Square) Matrices . . . . .	39
1.4.4	Positive Definite Matrices . . . . .	43
1.4.5	Eigenvalues, Eigenvectors, and Coordinate Trans- formations . . . . .	44

1.4.6	Schur Decomposition . . . . .	48
1.4.7	Singular Value Decomposition . . . . .	50
1.5	Functions of Matrices . . . . .	54
1.5.1	Polynomial and Exponential . . . . .	54
1.5.2	Optimizing Quadratic Functions . . . . .	58
1.5.3	Vec Operator and Kronecker Product of Matrices . . . . .	65
1.6	Exercises . . . . .	70
<b>2</b>	<b>Ordinary Differential Equations</b>	<b>99</b>
2.1	Introduction . . . . .	99
2.2	First-Order Linear Systems . . . . .	100
2.2.1	Superposition Principle for Linear Differential Equations . . . . .	100
2.2.2	Homogeneous Linear Systems with Constant Coefficients . . . . .	101
2.2.3	Qualitative Dynamics of Planar Systems . . . . .	104
2.2.4	Laplace Transform Methods for Solving the Inhomogeneous Constant-Coefficient Problem . . . . .	105
2.2.5	Delta Function . . . . .	113
2.3	Linear Equations with Variable Coefficients . . . . .	115
2.3.1	Introduction . . . . .	115
2.3.2	The Cauchy-Euler Equation . . . . .	115
2.3.3	Series Solutions and the Method of Frobenius . . . . .	116
2.4	Function Spaces and Differential Operators . . . . .	122
2.4.1	Functions as Vectors . . . . .	122
2.4.2	Self-Adjoint Differential Operators and Sturm-Liouville Equations . . . . .	129
2.4.3	Existence and Uniqueness of Solutions . . . . .	136
2.5	Lyapunov Functions and Stability . . . . .	148
2.5.1	Types of Stability . . . . .	148
2.5.2	Lyapunov Functions . . . . .	151
2.5.3	Application to Linear Systems . . . . .	156
2.5.4	Discrete Time Systems . . . . .	158
2.6	Asymptotic Analysis and Perturbation Methods . . . . .	161
2.6.1	Introduction . . . . .	161
2.6.2	Series Approximations: Convergence, Asymptoticness, Uniformity . . . . .	161
2.6.3	Scaling, and Regular and Singular Perturbations . . . . .	165
2.6.4	Regular Perturbation Analysis of an ODE . . . . .	168
2.6.5	Matched Asymptotic Expansions . . . . .	169

2.6.6	Method of Multiple Scales . . . . .	177
2.7	Qualitative Dynamics of Nonlinear Initial-Value Problems	182
2.7.1	Introduction . . . . .	182
2.7.2	Invariant Subspaces and Manifolds . . . . .	182
2.7.3	Some Special Nonlinear Systems . . . . .	186
2.7.4	Long-Time Behavior and Attractors . . . . .	190
2.7.5	The Fundamental Local Bifurcations of Steady States	196
2.8	Numerical Solutions of Initial-Value Problems . . . . .	203
2.8.1	Euler Methods: Accuracy and Stability . . . . .	204
2.8.2	Stability, Accuracy, and Stiff Systems . . . . .	207
2.8.3	Higher-Order Methods . . . . .	207
2.9	Numerical Solutions of Boundary-Value Problems . . . . .	210
2.9.1	The Method of Weighted Residuals . . . . .	210
2.10	Exercises . . . . .	223
<b>3</b>	<b>Vector Calculus and Partial Differential Equations</b>	<b>257</b>
3.1	Vector and Tensor Algebra . . . . .	257
3.1.1	Introduction . . . . .	257
3.1.2	Vectors in Three Physical Dimensions . . . . .	257
3.2	Differential Operators and Integral Theorems . . . . .	260
3.2.1	Divergence, Gradient, and Curl . . . . .	260
3.2.2	The Gradient Operator in Non-Cartesian Coordinates . . . . .	262
3.2.3	The Divergence Theorem . . . . .	268
3.2.4	Further Integral Relations and Adjoints of Multidimensional Differential Operators . . . . .	276
3.3	Linear PDEs: Properties and Solution Techniques . . . . .	278
3.3.1	Classification and Canonical Forms for Second-Order Partial Differential Equations . . . . .	278
3.3.2	Separation of Variables and Eigenfunction Expansion with Equations involving $\nabla^2$ . . . . .	279
3.3.3	Laplace's Equation, Spherical Harmonics, and the Hydrogen Atom . . . . .	294
3.3.4	Applications of the Fourier Transform to PDEs . . . . .	298
3.3.5	Green's Functions and Boundary-Value Problems	304
3.3.6	Characteristics and D'Alembert's Solution to the Wave Equation . . . . .	312
3.3.7	Laplace Transform Methods . . . . .	315
3.4	Numerical Solution of Initial-Boundary-Value Problems . . . . .	324

- 3.4.1 Numerical Stability Analysis for the Diffusion Equation . . . . . 325
- 3.4.2 Numerical Stability Analysis for the Convection Equation . . . . . 327
- 3.4.3 Operator Splitting for Convection-Diffusion Problems . . . . . 331
- 3.5 Exercises . . . . . 332
- 4 Probability, Random Variables, and Estimation 355**
- 4.1 Introduction and the Axioms of Probability . . . . . 355
- 4.2 Random Variables and the Probability Density Function 357
- 4.3 Multivariate Density Functions . . . . . 365
  - 4.3.1 Multivariate normal density . . . . . 367
  - 4.3.2 Functions of random variables. . . . . 376
  - 4.3.3 Statistical Independence and Correlation . . . . . 379
- 4.4 Sampling . . . . . 383
  - 4.4.1 Linear Transformation . . . . . 383
  - 4.4.2 Sample Mean, Sample Variance, and Standard Error 388
- 4.5 Central Limit Theorems . . . . . 390
  - 4.5.1 Identically distributed random variables . . . . . 391
  - 4.5.2 Random variables with different distributions . . 395
  - 4.5.3 Multidimensional central limit theorems . . . . . 396
- 4.6 Conditional Density Function and Bayes's Theorem . . . 397
- 4.7 Maximum-Likelihood Estimation . . . . . 401
  - 4.7.1 Scalar Measurement  $y$ , Known Measurement Variance  $\sigma^2$  . . . . . 402
  - 4.7.2 Scalar Measurement  $y$ , Unknown Measurement Variance  $\sigma^2$  . . . . . 409
  - 4.7.3 Vector of Measurements  $y$ , Different Parameters Corresponding to Different Measurements, Known Measurement Covariance  $R$  . . . . . 413
  - 4.7.4 Vector of Measurements  $y$ , Different Parameters Corresponding to Different Measurements, Unknown Measurement Covariance  $R$  . . . . . 420
  - 4.7.5 Vector of Measurements  $y$ , Same Parameters for all Measurements, Known Measurement Covariance  $R$  . . . . . 421
  - 4.7.6 Vector of Measurements  $y$ , Same Parameters for all Measurements, Unknown Measurement Covariance  $R$  . . . . . 424

4.8	PCA and PLS regression . . . . .	426
4.9	Appendix — Proof of the Central Limit Theorem . . . . .	434
4.10	Exercises . . . . .	440
<b>5</b>	<b>Stochastic Models and Processes</b>	<b>465</b>
5.1	Introduction . . . . .	465
5.2	Stochastic Processes for Continuous Random Variables . . . . .	466
5.2.1	Discrete Time Stochastic Processes . . . . .	466
5.2.2	Wiener Process and Brownian Motion . . . . .	469
5.2.3	Stochastic Differential Equations . . . . .	473
5.2.4	Fokker-Planck Equation . . . . .	480
5.3	Stochastic Kinetics . . . . .	485
5.3.1	Introduction, and Length and Time Scales . . . . .	485
5.3.2	Poisson Process . . . . .	487
5.3.3	Stochastic Simulation . . . . .	493
5.3.4	Master Equation of Chemical Kinetics . . . . .	496
5.3.5	Microscopic, Mesoscopic, and Macroscopic Kinetic Models . . . . .	502
5.4	Optimal Linear State Estimation . . . . .	508
5.4.1	Introduction . . . . .	508
5.4.2	Optimal Dynamic Estimator . . . . .	511
5.4.3	Optimal Steady-State Estimator . . . . .	516
5.4.4	Observability of a Linear System . . . . .	517
5.4.5	Stability of an Optimal Estimator . . . . .	520
5.5	Exercises . . . . .	523
<b>A</b>	<b>Mathematical Tables</b>	<b>538</b>
A.1	Laplace Transform Table . . . . .	538
A.2	Statistical Distributions . . . . .	541
A.3	Vector and Matrix Derivatives . . . . .	541
A.3.1	Derivatives: Other Conventions . . . . .	546
A.4	Exercises . . . . .	548
	<b>Author Index</b>	<b>550</b>
	<b>Citation Index</b>	<b>553</b>
	<b>Subject Index</b>	<b>555</b>